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# On The Order of Starlikeness and Strongly Starlikeness of Convex Functions of Order $\alpha$ and Strongly Convex of Order $\beta$

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Let  $\mathcal{A}$  denote the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the unit disc  $\mathbb{E} = \{z : |z| < 1\}$ . It is said to be starlike of order  $\alpha$ ,  $0 \leq \alpha < 1$ , if  $f(z) \in \mathcal{A}$  and

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad \text{in } \mathbb{E}.$$

We denote by  $\mathcal{S}_t(\alpha)$  this family of functions. It is said to be convex of order  $\alpha$ ,  $0 \leq \alpha < 1$ , if  $f(z) \in \mathcal{A}$  and

$$1 + \operatorname{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{in } \mathbb{E}.$$

We also denote by  $\mathcal{C}(\alpha)$  this family of functions.

A function  $f(z) \in \mathcal{A}$  is said to be strongly starlike of order  $\beta$ ,  $0 < \beta \leq 1$ , if

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{E}.$$

We denote this family of functions by  $\mathcal{SS}_t(\beta)$ . A function  $f(z) \in \mathcal{A}$  is said to be strongly convex of order  $\beta$ ,  $0 < \beta \leq 1$ , if

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{E}.$$

This family of functions is denoted by  $\mathcal{SC}(\beta)$ .

A.Marx [3] and E.Strohhäcker [5] showed that if  $f(z) \in \mathcal{C}(0)$  then  $f(z) \in \mathcal{S}_t\left(\frac{1}{2}\right)$ . It is well known that the number  $\frac{1}{2}$  is the largest value of  $\beta$  for which the assertion  $\mathcal{C}(0) \subset \mathcal{S}_t(\beta)$  holds, as is seen by the function  $f(z) = \frac{z}{1-z}$ .

I.S.Jack [1] posed the more general problem:

What is the largest number  $\beta(\alpha)$  so that  $\mathcal{C}(\alpha) \subset \mathcal{S}_t(\beta(\alpha))$ ?

Now, we introduce the new classes of starlike and convex functions. It is said to be

strongly starlike of order  $\beta$ ,  $0 < \beta \leq 1$ , and starlike of order  $\alpha$ ,  $0 \leq \alpha < 1$ , if  $f(z) \in \mathcal{A}$  and

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{E}.$$

We denote by  $\mathcal{SS}_i(\alpha, \beta)$  this family of functions.

On the other hand, it is said to be strongly convex of order  $\beta$ ,  $0 < \beta \leq 1$ , and convex of order  $\alpha$ ,  $0 \leq \alpha < 1$ , if  $f(z) \in \mathcal{A}$  and

$$\left| \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{E}.$$

This family of functions is also denoted by  $\mathcal{SC}(\alpha, \beta)$ . In [1], I.S.Jack obtained the following result.

**Theorem A.** If  $f(z) \in \mathcal{C}(\alpha)$ , then  $f(z) \in \mathcal{S}_i(\beta(\alpha))$ , where

$$\beta(\alpha) \geq \frac{2\alpha - 1 + \sqrt{9 - 4\alpha + 4\alpha^2}}{4}.$$

In [2], T.H.MacGregor claimed and conjectured the sharp result of  $\beta(\alpha)$  which improved Theorem A as the following.

**Theorem B.** If  $f(z) \in \mathcal{C}(\alpha)$ , then  $f(z) \in \mathcal{S}_i(\beta(\alpha))$ , where

$$\beta(\alpha) = \begin{cases} \frac{1 - 2\alpha}{2^{2-2\alpha}(1 - 2^{2\alpha-1})} & \text{if } \alpha \neq \frac{1}{2} \\ \frac{1}{2 \log 2} & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

In [6], D.R.Wilken and J.Feng completed the proof of Theorem B.

**Theorem 1.** If  $f(z) \in \mathcal{SC}(\alpha, n(\beta))$ , then  $f(z) \in \mathcal{SS}_i(\beta(\alpha), \beta)$ , where  $0 \leq \alpha < 1$ ,  $0 < \beta \leq 1$ ,

$$n(\beta) = \beta + \frac{2}{\pi} \tan^{-1} F(a_0),$$

$$F(a_0) = \min_{0 < a < \infty} F(a) = \min_{0 < a < \infty} \frac{G(a)}{H(a)},$$

$$G(a) = \frac{(a + a^{-1})}{2(a, \beta, l)} \left( a^\beta \beta \sin \left( \frac{\pi}{2}(1 - \beta) \right) + \beta l \right) - \frac{(\beta(\alpha) - \alpha)}{a^\beta} \sin \left( \frac{\pi}{2}\beta \right),$$

$$H(a) = (1 - \beta(\alpha)) + \frac{(a + a^{-1})}{2(a, \beta, l)} a^\beta \beta \cos \left( \frac{\pi}{2}(1 - \beta) \right) + \frac{(\beta(\alpha) - \alpha)}{a^\beta} \cos \left( \frac{\pi}{2}\beta \right),$$

$$l = \frac{\beta(\alpha)}{1 - \beta(\alpha)},$$

$$(a, \beta, l) = a^{2\beta} + 2a^\beta l \cos \left( \frac{\pi}{2}\beta \right) + l^2$$

and

$$\beta(\alpha) = \begin{cases} \frac{1 - 2\alpha}{2^{2-2\alpha}(1 - 2^{2\alpha-1})} & \text{if } \alpha \neq \frac{1}{2} \\ \frac{1}{2 \log 2} & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

*Proof* Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1,$$

and

$$q(z) = \frac{p(z) - \beta(\alpha)}{1 - \beta(\alpha)}, \quad q(0) = 1.$$

Then we have

$$p(z) = (1 - \beta(\alpha))q(z) + \beta(\alpha),$$

and

$$\frac{zp'(z)}{p(z)} = \frac{(1 - \beta(\alpha))zq'(z)}{(1 - \beta(\alpha))q(z) + \beta(\alpha)} = \left( \frac{zq'(z)}{q(z)} \right) \frac{q(z)}{q(z) + \frac{\beta(\alpha)}{1 - \beta(\alpha)}}.$$

Then it follows that

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} - \alpha &= p(z) + \frac{zp'(z)}{p(z)} - \alpha \\ &= (1 - \beta(\alpha))q(z) + \beta(\alpha) + \left( \frac{zq'(z)}{q(z)} \right) \frac{q(z)}{q(z) + \frac{\beta(\alpha)}{1 - \beta(\alpha)}} + \beta(\alpha) - \alpha \\ &= q(z) \left\{ (1 - \beta(\alpha)) + \left( \frac{zq'(z)}{q(z)} \right) \frac{q(z)}{q(z) + \frac{\beta(\alpha)}{1 - \beta(\alpha)}} + \frac{\beta(\alpha) - \alpha}{q(z)} \right\}. \end{aligned}$$

If there exists a point  $z_0 \in \mathbb{E}$  such that

$$|\arg q(z)| < \frac{\pi}{2}\beta \quad \text{for } |z| < |z_0|$$

and

$$|\arg q(z_0)| = \frac{\pi}{2}\beta,$$

then from [4], we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\beta$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg q(z_0) = \frac{\pi}{2}\beta$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg q(z_0) = -\frac{\pi}{2}\beta$$

where

$$q(z_0)^{\frac{1}{\beta}} = \pm ia, \quad \text{and} \quad a > 0.$$

At first, let us suppose

$$\arg q(z_0) = \frac{\pi}{2}\beta, \quad q(z_0) = (ia)^\beta,$$

and  $a > 0$ , then we have

$$\begin{aligned} & \arg \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \\ &= \arg q(z_0) + \arg \left\{ (1 - \beta(\alpha)) + \frac{i\beta k}{(ia)^\beta + \frac{\beta(\alpha)}{1-\beta(\alpha)}} + \left( \frac{\beta(\alpha) - \alpha}{a^\beta} \right) e^{-i\frac{\pi}{2}\beta} \right\} \\ &= \frac{\pi}{2}\beta + \arg \left\{ (1 - \beta(\alpha)) + \frac{\beta k e^{i\frac{\pi}{2}} (a^\beta e^{-i\frac{\pi}{2}\beta} + l)}{a^{2\beta} + 2a^\beta l \cos(\frac{\pi}{2}\beta) + l^2} + \left( \frac{\beta(\alpha) - \alpha}{a^\beta} \right) e^{-i\frac{\pi}{2}\beta} \right\} \\ &= \frac{\pi}{2}\beta + J \quad \text{say.} \end{aligned}$$

Then it follows that

$$\begin{aligned} J &\geq \arg \left\{ (1 - \beta(\alpha)) + \left( \frac{a + a^{-1}}{2(a, \beta, l)} \right) (a^\beta \beta e^{i\frac{\pi}{2}(1-\beta)} + i\beta l) + \left( \frac{\beta(\alpha) - \alpha}{a^\beta} \right) e^{-i\frac{\pi}{2}\beta} \right\} \\ &= \tan^{-1} \left\{ \frac{\left( \frac{a + a^{-1}}{2(a, \beta, l)} \right) (a^\beta \beta \sin(\frac{\pi}{2}(1-\beta)) + \beta l) - \left( \frac{\beta(\alpha) - \alpha}{a^\beta} \right) \sin(\frac{\pi}{2}\beta)}{(1 - \beta(\alpha)) + \left( \frac{a + a^{-1}}{2(a, \beta, l)} \right) a^\beta \beta \cos(\frac{\pi}{2}(1-\beta)) + \left( \frac{\beta(\alpha) - \alpha}{a^\beta} \right) \cos(\frac{\pi}{2}\beta)} \right\} \\ &= \tan_{0 < a < \infty}^{-1} \left( \frac{G(a)}{H(a)} \right) = \tan_{0 < a < \infty}^{-1} F(a) \geq \tan^{-1} F(a_0). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \arg \left( 1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right) &\geq \frac{\pi}{2}\beta + \tan^{-1} F(a_0) \\ &= \frac{\pi}{2}n(\beta). \end{aligned}$$

This contradicts the hypothesis of Theorem 1.

For the case  $\arg q(z_0) = -\frac{\pi}{2}\beta$ , applying the same method as the above, we can complete the proof of Theorem 1.

Putting  $\beta = 1$  in Theorem 1, we obtain T.H.MacGregor [3] and D.R.Wilken and J.Feng's result [6].

**Corollary 1.** If  $f(z) \in \mathcal{SC}(\alpha, n(1)) = \mathcal{SC}(\alpha, 1) = \mathcal{C}(\alpha)$ , then  $f(z) \in \mathcal{SS}_t(\beta(\alpha), 1) = \mathcal{S}_t(\beta(\alpha))$ .

*Proof* In the proof of Theorem 1, let us suppose that if there exists a point  $z_0 \in \mathbb{E}$  such that

$$|\arg q(z)| < \frac{\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg q(z_0)| = \frac{\pi}{2},$$

then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg q(z_0) = \frac{\pi}{2}$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg q(z_0) = -\frac{\pi}{2}$$

where  $q(z_0) = \pm ia$  and  $a > 0$ .

At first, let us suppose

$$\arg q(z_0) = \frac{\pi}{2}, \quad q(z_0) = ia$$

and  $a > 0$ , then we easily have  $H(a) > 0$ .

On the other hand, from A.Marx [3] and E.Strohhäcker's result [5], we have  $\beta \geq \frac{1}{2}$  for  $0 \leq \alpha < 1$ , therefore we have

$$\frac{\beta(\alpha)}{1 - \beta(\alpha)} = l \geq 1.$$

Then it follows that

$$\begin{aligned} G(a) &= \frac{(a + a^{-1})l}{2(a^2 + l^2)} - \frac{\beta(\alpha)}{a} \\ &= \frac{1}{2a} \left\{ \frac{(a^2 + l^2)l + l - l^3}{a^2 + l^2} - \beta(\alpha) + \alpha \right\} \\ &= \frac{1}{2a} \left\{ \frac{l(1 - l^2)}{a^2 + l^2} + l - \beta(\alpha) + \alpha \right\} \\ &> \frac{1}{2a} \left( \frac{l - l^3}{l^2} + l - \beta(\alpha) + \alpha \right) \\ &= \frac{1}{2a\beta(\alpha)} (1 - (1 - \alpha)\beta(\alpha) - \beta(\alpha)^2) \\ &= \frac{1}{2a\beta(\alpha)} Q(\alpha) \quad \text{say.} \end{aligned}$$

Now then,  $Q(\alpha)$  is a quadratic expression of  $\beta(\alpha)$ , the axis of parabolic curve is  $\frac{\alpha - 1}{2} < 0$ , this parabola opens downwards,

$$Q(0) = 1 - (1 - 0)\beta(0) - \beta(0)^2 = \frac{1}{4},$$

and

$$Q(1) = 1 - (1 - 0)\beta(1) - \beta(1)^2 = 1.$$

This shows that  $Q(\alpha) \geq 0$  for  $0 \leq \alpha < 1$ , therefore we have  $G(a) \geq 0$  for  $0 < a < \infty$  and it follows that

$$\lim_{a \rightarrow \infty} G(a) = 0.$$

Therefore, we have

$$\min_{0 < a < \infty} F(a) = \lim_{a \rightarrow \infty} \left( \frac{G(a)}{H(a)} \right) = 0,$$

and

$$n(1) = 1 + \frac{2}{\pi} \tan^{-1} F(a_0) = 1.$$

For the case,

$$\arg q(z_0) = -\frac{\pi}{2}, \quad q(z_0) = -ia, \quad a > 0,$$

applying the same method as the above, we can complete the proof of Corollary 1.

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